

PROFILE DECOMPOSITIONS AND BLOWUP PHENOMENA OF MASS CRITICAL FRACTIONAL SCHRÖDINGER EQUATIONS

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ABSTRACT. We study, under the radial symmetry assumption, the solutions to the fractional Schrödinger equations of critical nonlinearity in \mathbb{R}^{1+d} , $d \geq 2$, with Lévy index $2d/(2d-1) < \alpha < 2$. We firstly prove the linear profile decomposition and then apply it to investigate the properties of the blowup solutions of the nonlinear equations with mass-critical Hartree type nonlinearity.

1. INTRODUCTION

In [21] Laskin introduced the fractional quantum mechanics in which he generalized the Brownian-like quantum mechanical path, in the Feynman path integral approach to quantum mechanics, to the α -stable Lévy-like quantum mechanical path. This gives a rise to the fractional generalization of the Schrödinger equation. Namely, the associated equation for the wave function results in the fractional Schrödinger equations, which contains a nonlocal fractional derivative operator $(-\Delta)^{\frac{\alpha}{2}}$ defined by $(-\Delta)^{\frac{\alpha}{2}} = \mathcal{F}^{-1}|\xi|^{\alpha}\mathcal{F}$. In this paper we consider the following Cauchy problem with mass critical Hartree type nonlinearity:

$$(1.1) \quad \begin{cases} iu_t + (-\Delta)^{\frac{\alpha}{2}}u = \lambda(|x|^{-\alpha} * |u|^2)u, & (t, x) \in \mathbb{R}^{1+d}, d \geq 2, \\ u(0, x) = f(x) \in L^2, \end{cases}$$

where $\lambda = \pm 1$. Here α is Lévy stability index with $1 < \alpha \leq 2$. When $\alpha = 2$, the fractional Schrödinger equation becomes the well-known Schrödinger equation. See [22, 23] and references therein for further discussions related to the fractional quantum mechanics.

The solutions to equation (1.1) have the conservation laws for the mass and the energy:

$$M(u) = \int |u|^2 dx, \quad E(u) = \frac{1}{2} \int \overline{u} |\nabla|^{\alpha} u dx - \frac{\lambda}{4} \int \overline{u} (|x|^{-\alpha} * |u|^2) u dx.$$

We say that (1.1) is focusing if $\lambda = 1$, and defocusing if $\lambda = -1$. The equation (1.1) is mass-critical, as $M(u)$ is invariant under scaling symmetry $u_{\rho}(t, x) = \rho^{-d/2} u(t/\rho^{\alpha}, x/\rho)$, $\rho > 0$ which is again a solution to (1.1) with initial datum

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$\rho^{-d/2}u(0, x/\rho)$. The equation (1.1) is locally well-posed in L^2 for radial initial data and globally well-posed for sufficiently small radial data [7]. (See [14] for results regarding power type nonlinearities.) For the focusing case, the authors [8] used a virial argument to show the finite time blowup, with radial data, provided that the energy $E(u)$ is negative. Also see [12] and [13] for results with noncritical nonlinearity.

In this paper we aim to investigate the blowup phenomena of (1.1) with radial data when $\alpha < 2$. Due to the critical nonlinearity the time of existence no longer depends on the L^2 norm of initial data. Instead it relies on the profiles of the data. Hence the situation become more subtle. When $\alpha = 2$, a lot of work was devoted to the study of blowup phenomena, which was based on the usual Strichartz and its refinements. (See for instance [17, 19, 25].) When it comes to the fractional the Schrödinger equation ($1 < \alpha < 2$), due to the non-locality of fractional operator, various useful properties (e.g. Galilean invariance) which hold for the Schrödinger equation are no longer available. The main difficulty comes from absence of proper linear estimates. In fact, by scaling the condition $\alpha/q + d/r = d/2$ should be satisfied by the pair (q, r) if L^2 - $L_t^q L_x^r$ estimates were true for the linear propagator $f \rightarrow e^{it(-\Delta)^{\frac{\alpha}{2}}} f$. But such estimate is impossible as the Knapp example shows that \dot{H}^s - $L_t^q L_x^r$ is only possible for $2/q + d/r \leq d/2$. In order to get around these difficulties we work with radial assumption on the initial data, which allows us to use the recent results on the Strichartz estimates for radial functions [14] or angularly regular functions [9].

Linear profile decomposition. As for linear estimates such as the Strichartz estimates or Sobolev inequalities, the presence of noncompact symmetries causes defect of compactness. The profile decomposition with respect to the associated linear estimates is a measure to make it rigorous that such symmetries are the only source of non-compactness.

Concerning nonlinear dispersive equations (especially nonlinear wave and Schrödinger equations), the profile decompositions have been intensively studied and led to various recent developments in the study of equations with the critical nonlinearity ([17]). Profile decompositions for the Schrödinger equations with L^2 data were obtained by Merle and Vega [25] when $d = 2$, Carles and Keraani [5], $d = 1$, and Bégout and Vargas [3], $d \geq 3$. (Also see [1, 4, 28] for results on the wave equation and [29, 20, 10] on general dispersive equations.) These results are based on refinements of Strichartz estimates (see [26, 3]). There is a different approach based on Sobolev imbedding but such approach is not applicable especially the equation is L^2 -critical. Our approach is also based on a refinements of Strichartz estimate. Thanks to the extended range of admissible due to the radial assumption it is relatively simpler to obtain the refinement (see Proposition 2.3 which is used for the proof of profile decomposition.)

We now define the linear propagator $U(t)f$ to be the solution to the linear equation $iu_t + (-\Delta)^{\frac{\alpha}{2}}u = 0$ with initial datum f . Then it is formally given by

$$(1.2) \quad U(t)f = e^{it(-\Delta)^{\frac{\alpha}{2}}}f = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|^\alpha)} \widehat{f}(\xi) d\xi.$$

Here \widehat{f} denotes the Fourier transform of f such that $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$.

The following is our first result:

Theorem 1.1. *Let $d \geq 2$, $\frac{2d}{2d-1} < \alpha < 2$, and $2 < q, r < \infty$ satisfy $\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}$. Suppose that $(u_n)_{n \geq 1}$ is a sequence of complex-valued radial functions satisfying $\|u_n\|_{L_x^2} \leq 1$. Then up to a subsequence, for any $l \geq 1$, there exist a sequence of radial functions $(\phi^j)_{1 \leq j \leq l} \in L^2$, $\omega_n^l \in L^2$ and a family of parameters $(h_n^j, t_n^j)_{1 \leq j \leq l, n \geq 1}$ such that*

$$u_n(x) = \sum_{1 \leq j \leq l} U(t_n^j)[(h_n^j)^{-d/2} \phi^j(\cdot/h_n^j)](x) + \omega_n^l(x)$$

and the following properties are satisfied:

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|U_\alpha(\cdot) \omega_n^l\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} = 0,$$

and for $j \neq k$, $(h_n^j, t_n^j)_{n \geq 1}$ and $(h_n^k, t_n^k)_{n \geq 1}$ are asymptotically orthogonal in the sense that

$$\begin{aligned} & \text{either } \limsup_{n \rightarrow \infty} \left(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j} \right) = \infty, \\ & \text{or } (h_n^j) = (h_n^k) \text{ and } \limsup_{n \rightarrow \infty} \frac{|t_n^j - t_n^k|}{(h_n^j)^\alpha} = \infty, \end{aligned}$$

and for each l

$$\lim_{n \rightarrow \infty} \left[\|u_n\|_{L_x^2}^2 - \left(\sum_{1 \leq j \leq l} \|\phi^j\|_{L_x^2}^2 + \|\omega_n^l\|_{L^2}^2 \right) \right] = 0.$$

In what follows we make use of the linear profile decomposition to get nonlinear profile decompositions of the solutions to (1.1).

Nonlinear profile decomposition. Let us set

$$(q_\circ, r_\circ) = \left(3, \frac{6d}{3d-2\alpha} \right).$$

As it can be shown by the usual fixed point argument and the Strichartz estimate (with α -admissible pairs), in Lemma 2.1 the local well-posedness theory can be based on the estimate of space-time norm $\|u\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^1$. As a by product, if the solution fails to persist, then the space-time norm blows up.

Definition 1.2. *A solution $u \in C_t L_x^2((-T_{\min}, T_{\max}) \times \mathbb{R}^d)$ to (1.1) is said to blow up if $\|u\|_{L_t^{q_\circ} L_x^{r_\circ}((-T_{\min}, T_{\max}) \times \mathbb{R}^d)} = \infty$. Here $-T_{\min}, T_{\max} \in [-\infty, \infty]$ denote the maximal times of existence of the solution.*

¹In fact, one use choose any α -admissible (q, r) such that $6d/(3d-\alpha) \leq r \leq 6d/(3d-2\alpha)$ if $2d/(2d-1) < \alpha < 2$ and $6d/(3d-\alpha) < r \leq 6d/(3d-2\alpha)$ if $2d/(2d-1) = \alpha$. For instance see [6].

Since T_{max} or T_{min} may be ∞ , we regard non-scattering global solutions as blowup solutions at infinite time. We also define the minimal mass of solutions from which a solution may ignite to blow up.

Definition 1.3. $\delta_0 := \sup \{A \geq 0 : \text{If } \|u_0\|_{L_x^2} < A, \text{ for all } u_0 \in L_x^2 \text{ (1.1) is globally well-posed forward and backward, and its solution } u \text{ satisfies } \|u\|_{L_t^{q_0} L_x^{r_0}((-\infty, \infty) \times \mathbb{R}^d)} < \infty\}.$

By the small data global existence, we have $\delta_0 > 0$. Moreover, for any $\delta > \delta_0$, there exists a blowup solution u with $\delta_0 \leq \|u\|_{L^2} \leq \delta$. In Theorem 1.6 below we show that there exists a blowup solution having the minimal mass δ_0 . This will be shown by using the nonlinear profile decomposition, which is derived from the linear profile decomposition combined with perturbation theory.

For a given sequence of radial data $\{u_n^0\} \subset L_x^2$, from the linear profile decomposition (Theorem 1.1), we have an asymptotically orthogonal decomposition to a sequence $\{\phi^j\}_{1 \leq j \leq l} \in L^2$, $\omega_n^l \in L^2$, $(h_n^j, t_n^j)_{1 \leq j \leq l, n \geq 1}$. Then by taking subsequence, if necessary, we may assume that $t_n \in \{-\infty, 0, \infty\}$. Here we denote $t_n = \lim_j t_n^j$. Using the local well-posedness theorem with initial data at $t = 0$ or $t = \pm\infty$ (see Lemma A.1 below), we define the nonlinear profile by the maximal nonlinear solution for each linear profile.

Definition 1.4. Let $\{h_n, t_n\}$ be a family of parameters and $\{t_n\}$ have a limit in $[-\infty, \infty]$. Given a linear profile $\phi \in L_x^2$ with $\{(h_n, t_n)\}$, we define the nonlinear profile associated with it to be the maximal solution ψ to (1.1) which is in $C_t L_x^2((-\infty, \infty) \times \mathbb{R}^d)$ satisfying an asymptotic condition: For the sequence $\{t_n\}$,

$$\lim_{n \rightarrow \infty} \|U(t_n)\phi - \psi(t_n)\|_{L_x^2} = 0.$$

Then, the linear profile decomposition yields the nonlinear profile decomposition which is the key tool for proving blowup phenomena in what follows.

Proposition 1.5. Let $\{u_n^0\} \subset L_x^2$ be a bounded sequence. Suppose that $\{\phi^j\}_{1 \leq j \leq l} \in L^2$, $\omega_n^l \in L^2$, $(h_n^j, t_n^j)_{1 \leq j \leq l, n \geq 1}$ is a linear profile decomposition obtained from Theorem 1.1. Let $u_n \in C(J_n; L_x^2)$ be the maximal solution of (1.1) with initial data $u_n(t_0) = u_n^0$. For each $j \geq 1$, suppose $\{\psi^j\}_{1 \leq j \leq l} \in C_t L_x^2((-\infty, \infty) \times \mathbb{R}^d)$ is the maximal nonlinear profile associated with $\{\phi^j\}_{1 \leq j \leq l}$, $(h_n^j, t_n^j)_{1 \leq j \leq l, n \geq 1}$. Let $\{I_n\}$ be a family of nondecreasing time intervals containing 0. Then, the following two are equivalent;

- (1) $\|\Gamma_n^j \psi^j\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} < \infty, \quad j \geq 1,$
- (2) $\|u_n\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} < \infty.$

Here $\Gamma_n^j \psi^j = \frac{1}{(h_n^j)^{d/2}} \psi^j(\frac{t-t_n^j}{(h_n^j)^\alpha}, \frac{x}{h_n^j})$. Moreover, if (1) or (2) holds true, we have a decomposition

$$u_n = \sum_{j=1}^l \Gamma_n^j \psi^j + U(\cdot) \omega_n^l + e_n^l$$

with

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} (\|U(\cdot) \omega_n^l\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} + \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)}) = 0.$$

Blowup phenomena. We now consider the blowup solutions of (1.1) and present various results which rely on the nonlinear profile decomposition.

We first show the existence of minimal mass blowup solution. Due to lack of compactness of the Strichartz estimate, we do not expect that a bounded sequence has a convergent subsequence. However, the extremal sequence has a convergent subsequence and its limit. This can be viewed as a Palais-Smale type theorem (see [24]).

Theorem 1.6. *There exists a blowup solution u to (1.1) with initial data $f \in L^2$ of $\|f\|_{L^2} = \delta_0$. Moreover, $\{u(t) \in L^2 : -T_{\min} < t < T_{\max}\}$ is compact in L^2 modulo symmetries. That is, for any sequence $\{u(t_n)\}$ with $t_n \in (-T_{\min}, T_{\max})$, there exist $\phi \in L^2$ and a subsequence, still called $\{t_n\}$ and $\{h_n\}$, such that*

$$h_n^{d/2} u(t_n, h_n x) \rightarrow \phi \quad \text{in } L^2.$$

If the mass is greater than the minimal mass ($= \delta_0^2$) but less than twice of δ_0^2 , then the blowup solution does not form more than one blowup profile. Thus, we still have a weaker form of compactness property of the blowup solutions.

Theorem 1.7. *Let u be finite time blowup solution of (1.1) at T^* with $\|u(0, \cdot)\|_{L_x^2} < \sqrt{2} \delta_0$ and let $t_n \nearrow T^*$. Then there exist $\phi \in L_x^2$ and $\{h_n\}_{n=1}^\infty$ satisfying*

$$(1.3) \quad h_n^{d/2} u(t_n, h_n x) \rightharpoonup \phi \text{ weakly in } L_x^2$$

and if solution of (1.1) with initial data ϕ blows up at T^{**}

$$(1.4) \quad \limsup_{n \rightarrow \infty} \frac{h_n}{(T^* - t_n)^{1/\alpha}} \leq \frac{1}{(T^{**})^{1/\alpha}}.$$

Under the same condition, when a blowup occurs, only one profile blows up by shrinking in scale. As a corollary, we obtain the concentration in L^2 -norm at blowup time. For related results when $\alpha > 2$, see [6]. More precisely, we have

Corollary 1.8. *(Mass concentration of finite time blowup solution) Let u be a finite time blowup solution at T^* with $\|u(0, x)\|_{L^2} < \sqrt{2} \delta_0$ and let $t_n \nearrow T^*$. Then*

$$(1.5) \quad \limsup_{n \rightarrow \infty} \int_{|x| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \delta_0^2$$

for $\lambda(t_n)$ satisfying $\frac{(T^* - t_n)^{1/\alpha}}{\lambda(t_n)} \rightarrow 0$.

The arguments in this paper can be modified to prove the same results (non-linear profile decomposition and blowup phenomena) for the equations which have the power-type mass critical nonlinearities as long as we assume that blowup occurs. But the existence of blowup solutions does not seem to be known yet for the fractional Schrödinger equations with power-type nonlinearities.

The rest of the paper is organized as follows: In Section 2, we will show the refined Strichartz estimate. Section 3 will be devoted to establishing linear profile decomposition. Then we will show the nonlinear profile decomposition in Section 4. In Section 5 we will study blowup phenomena by making use of the profile decomposition.

2. REFINED STRICHARTZ ESTIMATES

It has been known that the Strichartz estimates for dispersive equations have wider admissible ranges when the initial data f are radial [14, 9]. Recently, almost optimal range of admissible pairs was established in [14] and the range was further extended in [9, 16] to include the remaining endpoint cases. We now recall from [9] that

$$(2.1) \quad \|U(\cdot)P_0f\|_{L_t^q L_x^r} \lesssim \|f\|_2$$

holds whenever $q, r \geq 2$, $(q, r) \neq (2, \frac{2(2d-1)}{2d-3})$, and $\frac{1}{q} \leq \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$.

Let $P_k, k \in \mathbb{Z}$, denote the Littlewood-Paley projection operator with symbol $\chi(\xi/2^k) \in C_0^\infty$ supported in the annulus $A_k = \{2^{k-1} < |\xi| \leq 2^{k+1}\}$ such that $\sum_{k \in \mathbb{Z}} P_k = id$. By (2.1), Littlewood-Paley decomposition and rescaling we get the following.

Lemma 2.1. *Let $\frac{2d}{2d-1} < \alpha < 2$, $q, r \geq 2$, and $r \neq \infty$, and let $\beta(\alpha, q, r) = d/2 - d/r - \alpha/q$. If $(q, r) \neq (2, \frac{2(2d-1)}{2d-3})$ and $\frac{1}{q} \leq \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$, then for radial f ,*

$$\|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sum_{k \in \mathbb{Z}} 2^{2k\beta(\alpha, q, r)} \|P_k f\|_2^2 \right)^{1/2}.$$

Adapting the argument of [6] together with Lemma 2.1, we get bilinear estimates for U which give extra smoothing due to interaction of two waves at different frequency levels.

Lemma 2.2. *Let $\frac{2d}{2d-1} < \alpha < 2$, $q, r > 2$, and $r \neq \infty$. Suppose that f and g are radial. Then, for $\frac{1}{q} < \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$, then there exists $\epsilon = \epsilon(\alpha, q, r) > 0$ such that*

$$\|U(\cdot)P_j f U(\cdot)P_k g\|_{L_t^{q/2} L_x^{r/2}} \lesssim 2^{(j+k)\beta(\alpha, q, r)} 2^{|j-k|\epsilon} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. By symmetry we may assume that $k \geq j$ and we set $\ell = j - k \leq 0$. Then, by rescaling it suffices to show, for some $\epsilon > 0$,

$$(2.2) \quad \|U(\cdot)P_\ell f U(\cdot)P_0 g\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim 2^{\ell(\beta(\alpha, q, r) + \epsilon)} \|f\|_2 \|g\|_2.$$

This estimate (2.2) with $\epsilon = 0$ obviously holds for $\frac{1}{q} \leq \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$, which follows from Lemma 2.1 and Hölder inequality. We can then interpolate this with the estimate

$$(2.3) \quad \|U(\cdot)P_\ell f U(\cdot)P_0 g\|_{L_{t,x}^2} \lesssim 2^{\ell(\beta(\alpha,4,4)+\epsilon)} \|f\|_2 \|g\|_2$$

for some $\epsilon > 0$ to get (2.2) for $\frac{1}{q} < \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$. Hence we reduce to showing (2.3).

When $2^\ell \sim 1$ (2.3) is trivial from Lemma 2.1. Thus we assume $2^\ell \ll 1$. By finite decomposition, rotation and a mild dilation, we also may assume that \widehat{g} is supported $\subset B(e_1, \epsilon)$. Here $e_1 = (1, 0, \dots, 0)$ and $B(e_1, \epsilon)$ is the ball of radius ϵ centered at e_1 . Freezing $\bar{\xi} = (\xi_2, \dots, \xi_d) \in B(0, 1)$, we set

$$B_{\bar{\xi}}(f, g)(x, t) = \frac{1}{(2\pi)^{2d}} \int e^{i(\xi+\eta)x+it(|\xi|^\alpha+|\eta|^\alpha)} \widehat{P_\ell f}(\xi) \widehat{P_0 g}(\eta) d\xi_1 d\eta.$$

Then it follows that

$$(2.4) \quad U(t)P_\ell f U(t)P_0 g = \int B_{\bar{\xi}}(f, g)(x, t) d\bar{\xi}.$$

We make change of variables $(\xi_1, \eta) \rightarrow \zeta = (\xi_1 + \eta_1, \dots, \xi_d + \eta_d, |\xi|^\alpha + |\eta|^\alpha)$ for $B_{\bar{\xi}}(f, g)(x, t)$. Then, by noting $\left| \frac{\partial(\zeta_1, \dots, \zeta_{d+1})}{\partial(\eta_1, \dots, \eta_d, \xi_1)} \right| = \alpha |\xi_1| |\xi|^{\alpha-2} - \eta_1 |\eta|^{\alpha-2}| \sim 1$, Plancherel's theorem, and reversing change of variables ($\zeta \rightarrow (\xi_1, \eta)$), we get

$$\|B_{\bar{\xi}}(f, g)\|_{L_{t,x}^2} \leq C \|\widehat{P_\ell f}(\xi) \widehat{P_0 g}(\eta)\|_{L_{\xi_1, \eta}^2}.$$

Therefore, by (2.4), Minkowski's inequality, and Hölder's inequality we get

$$\begin{aligned} \|U(\cdot)P_\ell f U(\cdot)P_0 g\|_{L_{t,x}^2} &\lesssim \int \|\widehat{P_\ell f}(\xi) \widehat{P_0 g}(\eta)\|_{L_{\xi_1, \eta}^2} d\bar{\xi} \\ &\lesssim \|g\|_2 \int \|\widehat{P_\ell f}(\xi)\|_{L_{\xi_1}^2 \chi_{\{|\bar{\xi}| < 2^{\ell+1}\}}} d\bar{\xi} \lesssim 2^{\ell(\frac{d-1}{2})} \|f\|_2 \|g\|_2. \end{aligned}$$

Since $\alpha \geq 0$ and $d \geq 2$, $\beta(\alpha, 4, 4) < \frac{d-1}{2}$. Hence, we can take $\epsilon = \frac{d-1}{2} - \beta(\alpha, 4, 4)$. This completes the proof of Lemma 2.2. \square

The estimate in Lemma 2.1 can be strengthened to get so-called *refinements of Strichartz estimate* ([3, 26, 27]). It plays crucial role in the proof of profile decomposition. Thanks to radial symmetry, such refinement is much easier to obtain. Here we make use of the argument in [6] where high order cases ($\alpha > 2$) were treated.

For $\alpha < 2$, let us call the pair (q, r) α -admissible, provided that $\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}$ for $2 \leq q, r \leq \infty$.

Proposition 2.3. *Let $\frac{2d}{2d-1} < \alpha < 2$, $q > 2$ and $r \neq \infty$. If (q, r) be α -admissible, then there exist θ, p , $\theta \in (0, 1)$, $1 \leq p < 2$, such that*

$$(2.5) \quad \|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sup_k 2^{kd(\frac{1}{2}-\frac{1}{p})} \|\widehat{P_k f}\|_p \right)^\theta \|f\|_2^{1-\theta}.$$

Proof. We have from Lemma 2.1

$$(2.6) \quad \|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sum_k \|\widehat{P_k f}\|_2^2 \right)^{1/2}$$

for any α -admissible pair (q, r) . Then (2.5) follows from interpolation of (2.6) and the following two estimates: for some p_*, q_* with $p_* < 2 < q_*$,

$$(2.7) \quad \|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sum_k \|\widehat{P_k f}\|_2^{q_*} \right)^{1/q_*},$$

$$(2.8) \quad \|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sum_k \left(2^{kd(\frac{1}{2} - \frac{1}{p_*})} \|\widehat{P_k f}\|_{p_*} \right)^2 \right)^{1/2}.$$

In fact, the interpolation among (2.6), (2.7) and (2.8) gives

$$(2.9) \quad \|U(\cdot)f\|_{L_t^q L_x^r} \lesssim \left(\sum_k \left(2^{kd(\frac{1}{2} - \frac{1}{p_0})} \|\widehat{P_k f}\|_{p_0} \right)^{q_0} \right)^{1/q_0}$$

for $(1/q_0, 1/p_0)$ on the triangle with the vertices $(1/2, 1/2)$, $(1/p_*, 1/2)$ and $(1/2, 1/q_*)$. So, there exist q_0, p_0 , $p_0 < 2 < q_0$, for which (2.9) holds. Hence,

$$\begin{aligned} \|U(\cdot)f\|_{L_t^q L_x^r} &\leq \left(\left(\sup_k 2^{kd(\frac{1}{2} - \frac{1}{p_0})} \|\widehat{P_k f}\|_{p_0} \right)^{q_0-2} \sum_k \left(2^{kd(\frac{1}{2} - \frac{1}{p_0})} \|\widehat{P_k f}\|_{p_0} \right)^2 \right)^{1/q_0} \\ &\leq \left(\sup_k 2^{kd(\frac{1}{2} - 1/p_0)} \|\widehat{P_k f}\|_{p_0} \right)^{(q_0-2)/q_0} \left(\sum_k \|\widehat{P_k f}\|_2^2 \right)^{1/q_0} \\ &\leq \left(\sup_k 2^{kd(\frac{1}{2} - \frac{1}{p_0})} \|\widehat{P_k f}\|_{p_0} \right)^{(q_0-2)/q_0} \|f\|_2^{2/q_0}. \end{aligned}$$

For the second inequality we used Hölder's inequality. We need only to set $p = p_0$ and $\theta = 1 - 2/q_0$ to get (2.5). Now we need to show (2.7) and (2.8).

We show (2.8) first. Let (q, r) , $2 < q < \infty$ be α -admissible. Since $\frac{2d}{2d-1} < \alpha$, there exist $2 < q_0, r_0 < \infty$ such that $\frac{n}{2} - \frac{n}{r_0} - \frac{\alpha}{q_0} > 0$, $\frac{1}{q_0} \leq \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r_0})$, and $(\frac{1}{q}, \frac{1}{r}) = \theta(\frac{1}{q_0}, \frac{1}{r_0})$, $0 < \theta < 1$. So, by (2.1) we have $\|U(\cdot)P_0 f\|_{L_t^{q_0} L_x^{r_0}} \lesssim \|\widehat{P_0 f}\|_2$. By interpolation of this with the trivial estimate $\|U(\cdot)P_0 f\|_{L_{t,x}^\infty} \lesssim \|\widehat{P_0 f}\|_1$, we get for some $1 < p_* < 2$, $\|U(\cdot)P_0 f\|_{L_t^q L_x^r} \lesssim \|\widehat{P_0 f}\|_{p_*}$. Then by rescaling, we have $\|U(\cdot)P_k f\|_{L_t^q L_x^r} \lesssim 2^{kd(\frac{1}{2} - \frac{1}{p_*})} \|\widehat{P_0 f}\|_{p_*}$. Now Littlewood-Paley and Minkowski inequalities give

$$\|U(\cdot)f\|_{L_t^q L_x^r} \leq \left(\sum_k \|U(\cdot)P_k f\|_{L_t^q L_x^r}^2 \right)^{1/2} \lesssim \left(\sum_k \left(2^{kd(\frac{1}{2} - \frac{1}{p_*})} \|\widehat{P_k f}\|_{p_*} \right)^2 \right)^{1/2}.$$

We now turn to the proof of (2.7). It is sufficient to show an $L_{t,x}^4$ estimate

$$(2.10) \quad \|U(\cdot)f\|_{L_{t,x}^4} \lesssim \left(\sum_k \left(2^{k\beta(\alpha, 4, 4)} \|\widehat{P_k f}\|_2 \right)^4 \right)^{\frac{1}{4}}.$$

Indeed, as before, the required estimates can be obtained by interpolating (2.10) with the estimates in Lemma 2.1 for $(q, r) \neq (2, \frac{2(2d-1)}{2d-3})$ which satisfy $\frac{1}{\alpha}(\frac{1}{2} - \frac{1}{r}) <$

$\frac{1}{q} \leq \frac{2d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right)$, $2 \leq q, r \leq \infty$. To show (2.10) we write

$$U(t)f U(t)f = \sum_{j=-\infty}^{\infty} \sum_k U(t)P_k f U(t)P_{j+k} f.$$

For (2.10) it is sufficient to show that for some $\epsilon > 0$

$$(2.11) \quad \left\| \sum_k U(\cdot)P_k f U(\cdot)P_{j+k} f \right\|_{L^2_{t,x}} \lesssim 2^{-|j|\epsilon} \left(\sum_k (2^{k\beta(\alpha,4,4)} \|\widehat{P_k f}\|_2)^4 \right)^{1/4}.$$

We show it by considering the cases $|j| \leq 3$ and $|j| > 3$, separately. Let us first consider the case $|j| \leq 3$. By Cauchy-Schwartz inequality, $|\sum_k U(t)P_k f U(t)P_{j+k} f|^2 \leq \sum_{l=-\infty}^{\infty} \sum_k |U(t)P_k f U(t)P_{k+l} f|^2$. So,

$$\left\| \sum_k U(t)P_k f U(t)P_{j+k} f \right\|_{L^2_{t,x}}^2 \leq \sum_{l=-\infty}^{\infty} \sum_k \|U(t)P_k f U(t)P_{k+l} f\|_{L^2_{t,x}}^2.$$

From Lemma 2.2 and Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} \left\| \sum_k U(t)P_k f U(t)P_{j+k} f \right\|_{L^2_{t,x}}^2 &\lesssim \sum_{l=-\infty}^{\infty} \sum_k 2^{-\epsilon|l|} 2^{2k\beta(\alpha,4,4)} \|P_k f\|_2^2 2^{2(k+l)\beta(\alpha,4,4)} \|P_{k+l} f\|_2^2 \\ &\leq \sum_{l=-\infty}^{\infty} 2^{-\epsilon|l|} \sum_k \left(2^{k\beta(\alpha,4,4)} \|P_k f\|_2^4 \right)^4 \lesssim \sum_k \left(2^{k\beta(\alpha,4,4)} \|P_k f\|_2^4 \right)^4 \end{aligned}$$

for $|j| \leq 3$. We now consider the case $|j| > 3$. Let us first observe that the Fourier supports of $U(t)P_k f U(t)P_{k+j} f$ are boundedly overlapping. So by Plancherel's theorem and Lemma 2.2

$$\begin{aligned} \left\| \sum_k U(\cdot)P_k f U(\cdot)P_{k+j} f \right\|_{L^2_{t,x}}^2 &\lesssim \sum_k \|U(\cdot)P_k f U(\cdot)P_{k+j} f\|_{L^2_{t,x}}^2 \lesssim \\ &\sum_k 2^{-|j|\epsilon} 2^{k\beta(\alpha,4,4)} \|P_k f\|_2^2 2^{(k+j)\beta(\alpha,4,4)} \|P_{k+j} f\|_2^2 \lesssim 2^{-|j|\epsilon} \sum_k (2^{k\beta(\alpha,4,4)} \|P_k f\|_2^2)^4. \end{aligned}$$

This completes the proof. \square

3. LINEAR PROFILE DECOMPOSITION

In this section we prove Theorem 1.1. We assume that the pair (q, r) is α -admissible with $\frac{d}{2}(\frac{1}{2} - \frac{1}{r}) < \frac{1}{q} < \frac{2n-1}{2}(\frac{1}{2} - \frac{1}{r})$, that is, $\frac{2d}{2d-1} < \alpha < 2$.

3.1. Preliminary decomposition. By using the refined Strichartz estimate (2.5), we extract frequencies and scaling parameters to get a preliminary decomposition as follows.

Proposition 3.1. *Let $(u_n)_{n \geq 1}$ be a sequence of complex valued functions with $\|u_n\|_{L^2} \leq 1$. Then for any $\delta > 0$, there exists $N = N(\delta)$, $\rho_n^j \in (0, \infty)$ and $(f_n^j)_{1 \leq j \leq N, n \geq 1} \subset L^2$ such that*

$$u_n = \sum_{j=1}^N f_n^j + q_n^N$$

with the following properties:

- 1) there exist compact set $K = K(N)$ in an annulus $\{\xi : R_1 < |\xi| < R_2\}$ satisfying that

$$(\rho_n^j)^{d/2} |\widehat{f_n^j}(\rho_n^j \xi)| \leq C_\delta \chi_K(\xi) \text{ for every } 1 \leq j \leq N,$$

- 2) $\limsup_{n \rightarrow \infty} (\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j}) = \infty$, if $j \neq k$,
- 3) $\limsup_{n \rightarrow \infty} \|U(\cdot) q_n^N\|_{L_t^q L_x^r} \leq \delta$ for any $N \geq 1$,
- 4) $\limsup_{n \rightarrow \infty} (\|u_n\|_2^2 - (\sum_{j=1}^N \|f_n^j\|_2^2 + \|q_n^N\|_2^2)) = 0$.

Proof. Suppose that $\limsup_{n \rightarrow \infty} \|U(\cdot) u_n\|_{L_t^q L_x^r} \leq \delta$, then there is nothing to prove. So we assume that $\|U(\cdot) u_n\|_{L_t^q L_x^r} > \delta$ for all $n \geq 1$. By refined Strichartz estimates (Lemma 2.3), there exists $A_n^1 = \{\xi : \rho_n^1/2 < |\xi| < \rho_n^1\}$ such that

$$c_1 (\rho_n^1)^{d(\frac{1}{p}-\frac{1}{2})} \delta^{\frac{1}{\theta}} \leq \|\widehat{u_n^1}\|_p \text{ for some constant } c_1,$$

where $\widehat{u_n^1} = \widehat{u_n} \chi_{A_n^1}$. And for any $\lambda > 0$

$$\int_{\{|\widehat{u_n^1}| > \lambda\}} |\widehat{u_n^1}|^p d\xi = \int_{\{|\widehat{u_n^1}| > \lambda\}} (\lambda^{2-p} |\widehat{u_n^1}|^p) \lambda^{p-2} d\xi \leq \lambda^{p-2}.$$

Thus we have

$$\left(\int_{\{|\widehat{u_n^1}| > \lambda\}} |\widehat{u_n^1}|^p d\xi \right)^{\frac{1}{p}} \leq \lambda^{1-\frac{2}{p}}.$$

Let $\lambda = (\frac{c_1}{2})^{\frac{p}{2-p}} (\rho_n^1)^{-\frac{d}{2}} \delta^{\frac{1}{\theta} \cdot \frac{p}{p-2}}$. Then

$$\frac{c_1}{2} (\rho_n^1)^{d(\frac{1}{p}-\frac{1}{2})} \delta^{\frac{1}{\theta}} \leq \left(\int_{\{|\widehat{u_n^1}| < \lambda\}} |\widehat{u_n^1}|^p d\xi \right)^{\frac{1}{p}} \leq (\omega_d^{\frac{1}{2}} \rho_n^1)^{d(\frac{1}{p}-\frac{1}{2})} \left(\int_{\{|\widehat{u_n^1}| < \lambda\}} |\widehat{u_n^1}|^2 d\xi \right)^{\frac{1}{2}},$$

where ω_d is the measure of unit sphere, which implies that

$$\frac{c_1'}{2} \delta^{\frac{1}{\theta}} \leq \left(\int_{\{|\widehat{u_n^1}| < \lambda\}} |\widehat{u_n^1}|^2 d\xi \right)^{\frac{1}{2}} \quad (c_1' = c_1 \omega_d^{1/2-1/p}).$$

Now define $G_n^1(\psi)(\xi)$ by $(\rho_n^1)^{d/2} \psi(\rho_n^1 \xi)$ for measurable function ψ . Then by letting $\widehat{v_n^1} = \widehat{u_n^1} \chi_{\{|\widehat{u_n^1}| < \lambda\}}$ we get $\|v_n^1\|_2 \geq \frac{1}{2} c_1' \delta^{\frac{1}{\theta}}$ and $|G_n^1(\widehat{v_n^1})(\xi)| = (\rho_n^1)^{\frac{d}{2}} \widehat{v_n^1}(\rho_n^1 \xi) \leq C_\delta \chi_{A_{R_1, R_2}}(\xi)$, where A_{R_1, R_2} is the annulus $\{\xi : R_1 < |\xi| < R_2\}$. We can repeat above progress with $u_n - v_n^1$ replacing u_n . After $N(=N(\delta))$ steps², we get $(v_n^j)_{1 \leq j \leq N}$ and (ρ_n^j) such that

$$\begin{aligned} u_n &= \sum_{j=1}^N v_n^j + q_n^N, \\ \|u_n\|_2^2 &= \sum_{j=1}^N \|v_n^j\|_2^2 + \|q_n^N\|_2^2, \\ \|U(t) q_n^N\|_{L_{t,x}^r} &\leq \delta. \end{aligned}$$

²At each step, the L^2 norm decreases by at least $\frac{1}{2} c_1' \delta^{\frac{1}{\theta}}$.

The second identity follows from disjointness of ³ Fourier supports of v_n^j and q_n^N . The third inequality gives Property 3).

We say $\rho_n^j \perp \rho_n^k$ if and only if $\limsup (\frac{\rho_n^k}{\rho_n^j} + \frac{\rho_n^j}{\rho_n^k}) = \infty$. Define f_n^1 to be the sum of those v_n^j whose ρ_n^j are not orthogonal to ρ_n^1 . Take least $j_0 \in [2, N]$ such that $\rho_n^{j_0}$ is orthogonal to ρ_n^1 and define f_n^2 to be the sum of v_n^j whose ρ_n^j are orthogonal to ρ_n^1 but not to $\rho_n^{j_0}$. After finite step, we have $(f_n^j)_{1 \leq j \leq N}$ satisfying properties 2) and 4) because the Fourier supports are disjoint.

Now we have only to check Property 1). We only consider f_n^1 . The other cases can be treated similarly. Since v_n^j collected in f_n^1 has ρ_n^j which is not orthogonal to ρ_n^1 , we have

$$(3.1) \quad \limsup_{n \rightarrow \infty} \left(\frac{\rho_n^j}{\rho_n^1} + \frac{\rho_n^1}{\rho_n^k} \right) < \infty.$$

And by construction, we also have $|G_n^j(\widehat{v_n^j})| \leq C_\delta \chi_{A_{1/2,1}}$. Here $G_n^j(\psi)(\xi) = (\rho_n^j)^{\frac{d}{2}} \psi(\rho_n^j \xi)$. Since $G_n^1(\widehat{v_n^j}) = G_n^1(G_n^j)^{-1} G_n^j(\widehat{v_n^j})$ and

$$G_n^1(G_n^j)^{-1} \psi(\xi) = \left(\frac{\rho_n^1}{\rho_n^j} \right)^{\frac{d}{2}} \psi \left(\frac{\rho_n^1}{\rho_n^j} \xi \right),$$

from the non-orthogonality (3.1) it follows that there exist R_1 and R_2 with $0 < R_1 < R_2$ such that $|G_n^1(\widehat{v_n^j})| \leq \widetilde{C}_\delta \chi_{A_{R_1, R_2}}$ for all v_n^j collected in f_n^1 . This completes the proof of Proposition 3.1. \square

The next step is devoted to further decomposition of f_n^j to get time parameters.

Proposition 3.2. *Suppose that $\{f_n\} \subset L^2$ satisfies $(\rho_n)^{d/2} |\widehat{f_n}(\rho_n \xi)| \leq \widehat{F}(\xi)$ and $\widehat{F} \in L^\infty(K)$ for some compact set $K \subset A = \{\xi : 0 < R_1 < |\xi| < R_2\}$. Then there exist a family $(s_n^\ell)_{\ell \geq 1} \subset \mathbb{R}$ and a sequence $(\phi^\ell)_{\ell \geq 1} \subset L^2$ satisfying the following properties:*

1) for $\ell \neq \ell'$

$$\limsup_{n \rightarrow \infty} |s_n^\ell - s_n^{\ell'}| = \infty,$$

2) for every $M \geq 1$, there exists $e_n^M \in L^2$ such that

$$f_n(x) = \sum_{\ell=1}^M (\rho_n)^{d/2} (U(s_n^\ell) \phi^\ell)(\rho_n x) + e_n^M(x)$$

and

$$\limsup_{\substack{M \rightarrow \infty \\ n \rightarrow \infty}} \|U(\cdot) e_n^M\|_{L_t^q L_x^r} = 0,$$

3) for any $M \geq 1$,

$$\limsup_{n \rightarrow \infty} \left(\|f_n\|^2 - \left(\sum_{\ell=1}^M \|\phi^\ell\|_2^2 + \|e_n^M\|_2^2 \right) \right) = 0.$$

³Actually, we can make them mutually disjoint at each step.

Proof. Let us denote by \mathcal{F} the collection of functions $\{F_n\}_{n \geq 1}$ which are given by $\widehat{F_n}(\xi) = (\rho_n)^{d/2} \widehat{f_n}(\rho_n \xi)$, and define

$$\mathcal{W}(\mathcal{F}) = \{\text{weak-lim } U(-s_n^1)F_n \text{ in } L^2 : s_n^1 \in \mathbb{R}\}$$

and $\mu(\mathcal{F}) = \sup_{\phi \in \mathcal{W}(\mathcal{F})} \|\phi\|_{L^2}$. Then $\mu(\mathcal{F}) \leq \limsup_{n \rightarrow \infty} \|F_n\|_{L^2}$.

We may assume that $\mu(\mathcal{F}) > 0$. As a matter of fact, if $\mu(\mathcal{F}) = 0$, we are done because we will show later (3.2) for some θ , $0 < \theta < 1$.

Let us choose a subsequence $\{F_n\}$, s_n^1 and ϕ^1 such that $U(-s_n^1)F_n \rightharpoonup \phi^1$ as $n \rightarrow \infty$ and $\|\phi^1\| \geq \frac{1}{2}\mu(\mathcal{F})$. Let $F_n^1 = F_n - U(s_n^1)\phi^1$ and $\mathcal{F}^1 = \{F_n^1\}$. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|F_n^1\|_2^2 &= \limsup_{n \rightarrow \infty} \langle F_n - U(s_n^1)\phi^1, F_n - U(s_n^1)\phi^1 \rangle \\ &= \limsup_{n \rightarrow \infty} \langle U(-s_n^1)F_n - \phi, U(-s_n^1)F_n - \phi^1 \rangle \\ &= \limsup_{n \rightarrow \infty} (\langle F_n, F_n \rangle - \langle U(-s_n^1)F_n, \phi^1 \rangle - \langle \phi^1, U(-s_n^1)F_n \rangle + \langle \phi^1, \phi^1 \rangle) \\ &= \limsup_{n \rightarrow \infty} \|F_n\|_2^2 - \|\phi^1\|_2^2. \end{aligned}$$

Repeat the process with F_n^1 to get s_n^2, ϕ^2, F_n^2 and so on. By taking a diagonal sequence we may write

$$F_n(x) = \sum_{\ell=1}^M U(s_n^\ell)\phi^\ell + F_n^M,$$

which satisfies that $\limsup_{n \rightarrow \infty} \|F_n\|_2^2 = \sum_{\ell=1}^M \|\phi^\ell\|_2^2 + \limsup_{n \rightarrow \infty} \|F_n^M\|_2^2$. So $\sum_{\ell=1}^M \|\phi^\ell\|_2^2$ is convergent, which implies $\limsup_{\ell \rightarrow \infty} \|\phi^\ell\|_2 = 0$. Since $\mu(\mathcal{F}^M) \leq 2\|\phi^{M+1}\|_2$, we get $\limsup_{M \rightarrow \infty} \mu(\mathcal{F}^M) = 0$.

Now let us define e_n^M by $\widehat{F_n^M} = \rho_n^{\frac{d}{2}} \widehat{e_n^M}$. Then the remaining thing is to show

$$(3.2) \quad \limsup_{n \rightarrow \infty} \|U(\cdot)e_n^M\|_{L_t^q L_x^r} \lesssim \mu(\mathcal{F}^M)^\theta$$

for some θ with $0 < \theta < 1$. By construction, we may assume $\widehat{\phi}^\ell_{1 \leq \ell \leq M}$ has common compact support K . Invoking that the pair (q, r) is α -admissible with $\frac{1}{q} < \frac{2d-1}{2}(\frac{1}{2} - \frac{1}{r})$, we get

$$\|U(\cdot)e_n^M\|_{L_t^q L_x^r} = \|U(\cdot)F_n^M\|_{L_t^q L_x^r} \leq \|U(\cdot)F_n^M\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}}^{\tilde{q}/q} \|U(\cdot)F_n^M\|_{L_{t,x}^\infty}^{1-\tilde{q}/q}$$

for some $\frac{2d}{2d-1}$ -admissible pair (\tilde{q}, \tilde{r}) with $\frac{\tilde{q}}{q} = \frac{\tilde{r}}{r}$. Concerning the first term, from Lemma 2.1 we have

$$\|U(\cdot)F_n^M\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \lesssim R_1^{\frac{d}{2} - \frac{2d}{q(2d-a)} - \frac{d}{r}} \|F_n^M\|_{L^2} \lesssim R_1^{\frac{d}{2} - \frac{2d}{q(2d-a)} - \frac{d}{r}}.$$

Thus for (3.2) it suffices to show $\limsup_{n \rightarrow \infty} \|U(t)F_n^M\|_{L_{t,x}^\infty} \lesssim \mu(\mathcal{F}^M)$. For this we may assume that there exists $\delta > 0$ such that

$$\limsup_{\substack{M \rightarrow \infty \\ n \rightarrow \infty}} \|U(t)F_n^M\|_{L_{t,x}^\infty} > \delta.$$

Let (s_n^M, y_n^M) be such that $\|U(t)F_n^M\|_{L_{t,x}^\infty} = |U(s_n^M)(F_n^M)(y_n^M)|$. Then we show that $|y_n^M|$ is uniformly bounded.

Let us first observe that for any $x_0, x_1 \in \mathbb{R}^d$

$$\begin{aligned} |U(t)F_n^M(x_1) - U(t)F_n^M(x_0)| &\leq \sup_x |\nabla(U(t)F_n^M(x))| |x_1 - x_0| \\ &\leq \int |\xi| |e^{it|\xi|^\alpha} \widehat{P_n^M}(\xi)| d\xi |x_1 - x_0| \lesssim \left(\int_0^{R_2} r^2 \cdot r^{n-1} dr \right)^{\frac{1}{2}} \|F_n^M\|_2 |x_1 - x_0| \\ &\lesssim R_2^{\frac{n+2}{2}} |x_1 - x_0|. \end{aligned}$$

From this, we deduce that $|U(s_n^M)F_n^M(y)| > \frac{\delta}{2}$ if $|y - y_n^M| \leq c\frac{\delta}{2}$ for some small constant $c > 0$. Since $U(s_n^M)(F_n^M)(y)$ is radially symmetric,

$$|U(s_n^M)(F_n^M)(y)| > \frac{\delta}{2} \quad \text{if} \quad |y_n^M| - c\frac{\delta}{2} < |y| < |y_n^M| + c\frac{\delta}{2}.$$

Taking L^2 norm on $|y_n^M| - c\frac{\delta}{2} < |y| < |y_n^M| + c\frac{\delta}{2}$, we have $\frac{\delta}{2}|y_n^M|^{n-1}\frac{\delta}{2}c \leq \|F_n^M\|_2 \leq 1$, which implies $|y_n^M|$ is uniformly bounded. Since $|y_n^M|$ is uniformly bounded, there exists y_0^M such that $y_n^M \rightarrow y_0^M$ as $n \rightarrow \infty$ for some subsequence. Then for large n , $|U(s_n^M)(F_n^M)(y_0^M)| \geq \frac{1}{2}|U(s_n^M)(F_n^M)(y_n^M)|$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be radially symmetric and such that $\psi = 1$ on K . And let ψ^M be Schwartz function such that $\widehat{\psi^M} = \psi \widehat{\delta_{y_0^M}}$, where $\delta_{y_0^M}$ is Dirac-delta measure. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|U(t)F_n^M\|_{L_{t,x}^\infty} &\lesssim \limsup_{n \rightarrow \infty} |U(s_n^M)(F_n^M)(y_0^M)| \\ &= \limsup_{n \rightarrow \infty} \left| \int U(s_n^M)(F_n^M)(y) \psi^M(y) dy \right| \\ &\leq \|\psi^M\|_2 \mu(\mathcal{F}^M) \lesssim \mu(\mathcal{F}^M). \end{aligned}$$

This completes the proof of Proposition 3.2. \square

We now begin the proof of Theorem 1.1.

3.2. Proof of Theorem 1.1. Let us start with the preliminary decomposition. From Propositions 3.1 and 3.2 we have

$$(3.3) \quad u_n = \sum_{j=1}^N \sum_{\ell=1}^{M_j} \Phi_n^{\ell,j} + \omega_n^{N, M_1, \dots, M_N},$$

where

$$\begin{aligned} \Phi_n^{\ell,j} &= U(t_n^{\ell,j})[(h_n^j)^{-d/2} \phi^{\ell,j}(\cdot/h_n^j)], \\ (h_n^j, t_n^{\ell,j}) &= ((\rho_n^j)^{-1}, (\rho_n^j)^{-\alpha} s_n^{\ell,j}), \quad \omega_n^{N, M_1, \dots, M_N} = \sum_{j=1}^N e_n^{j, M_j} + q_n^N. \end{aligned}$$

Then the decomposition satisfies

- (1) by constructions, the family $(h_n^j, t_n^{\ell,j})$ is pairwise orthogonal,

(2) the asymptotic orthogonality is satisfied as follows:

$$\|u_n\|_2^2 = \sum_{j=1}^N \sum_{\ell=1}^{M_j} \|\phi^{\ell,j}\|_2^2 + \|\omega_n^{N,M_1,\dots,M_N}\|_2^2 + o_n(1)$$

and $\|\omega_n^{N,M_1,\dots,M_N}\|_2^2 = \sum_{j=1}^N \|e_n^{j,M_j}\|_2^2 + \|q_n^N\|_2^2$ due to disjoint Fourier supports.

We will show that $U(t)\omega_n^{N,M_1,\dots,M_N}$ converges to zero in a Strichartz norm, i.e.,

$$(3.4) \quad \limsup_{n \rightarrow \infty} \|U(t)\omega_n^{N,M_1,\dots,M_N}\|_{L_t^q L_x^r} \rightarrow 0 \text{ as } \min\{N, M_1, \dots, M_N\} \rightarrow \infty,$$

where (q, r) is an α -admissible pair with $\frac{2d}{2d-1} < \alpha < 2$. We enumerate the pair (j, α) by v satisfying

$$v(j, \alpha) < v(k, \beta) \text{ if } j + \alpha < k + \beta \text{ or } j + \alpha = k + \beta \text{ and } j < k.$$

After relabeling,

$$u_n = \sum_{1 \leq j \leq l} U(t_n^j) [(h_n^j)^{-d/2} \phi^j(\cdot/h_n^j)] + \omega_n^l$$

where $\omega_n^j = u_M^{N,M_1,\dots,M_n}$ with $l = \sum_{j=1}^N M_j$. Then the proof is completed by (3.4).

Now let us prove (3.4). Given $\varepsilon > 0$, we take a positive number Λ such that for every $N \geq \Lambda$,

$$\limsup_{n \rightarrow \infty} \|U(t)q_n^N\|_{L_t^q L_x^r} \leq \varepsilon/3$$

Then for every $N \geq \Lambda$, we can find Λ_N such that whenever $M_j \geq \Lambda_N$,

$$\limsup_{n \rightarrow \infty} \|U(t)e_n^{j,M_j}\|_{L_t^q L_x^r} \leq \varepsilon/3N.$$

Now we rewrite $\omega_n^{N,M_1,\dots,M_N}$ by

$$\omega_n^{N,M_1,\dots,M_N} = q_n^M + \sum_{1 \leq j \leq N} e_n^{j,M_j \vee \Lambda_N} + R_n^{N,M_1,\dots,M_N},$$

where $M_j \vee \Lambda_N$ denotes $\max\{M_j, \Lambda_N\}$ and

$$R_n^{N,M_1,\dots,M_N} = \sum_{1 \leq j \leq N} (e_n^{j,M_j} - e_n^{j,\Lambda_N}) = \sum_{\substack{1 \leq j \leq N \\ M_j < \Lambda_N}} \sum_{M_j < \ell < \Lambda_N} \Phi_n^{\ell,j}.$$

Then we have

$$\lim_{n \rightarrow \infty} \|U(t)\omega_n^{N,M_1,\dots,M_N}\|_{L_t^q L_x^r} \leq \frac{2\varepsilon}{3} + \lim_{n \rightarrow \infty} \|U(t)R_n^{N,M_1,\dots,M_N}\|_{L_t^q L_x^r}.$$

In order to handle last term, we need the following lemma which will be proved at the end of this section.

Lemma 3.3. *For every N, M_1, \dots, M_N , we have*

$$(3.5) \quad \limsup_{n \rightarrow \infty} \left\| \sum_{j=1}^N \sum_{\ell=1}^{M_j} U(t)\Phi_n^{\ell,j} \right\|_{L_t^q L_x^r}^2 \leq \sum_{j=1}^N \sum_{\ell=1}^{M_j} \limsup_{n \rightarrow \infty} \|U(t)\Phi_n^{\ell,j}\|_{L_t^q L_x^r}^2.$$

From Lemma 3.3 and Strichartz estimates (Lemma 2.1) it follows that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|U(t)R_n^{N,M_1,\dots,M_N}\|_{L_t^q L_x^r}^2 &\leq \sum_{\substack{1 \leq j \leq N \\ M_j < \Lambda_N}} \sum_{M_j < \ell < \Lambda_N} \limsup_{n \rightarrow \infty} \|U(t)\Phi_n^{\ell,j}\|_{L_t^q L_x^r}^2 \\ &\lesssim \sum_{1 \leq j \leq N} \sum_{\ell > M_j} \|\phi^{\ell,j}\|_2^2. \end{aligned}$$

Since $\sum_{j,\ell} \|\phi^{\ell,j}\|_2^2$ is convergent,

$$\limsup_{n \rightarrow \infty} \left(\sum_{j=1}^N \sum_{\ell > M_j} \|U(t)\Phi_n^{\ell,j}\|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}} \leq \frac{\varepsilon}{3},$$

provided that $\min(N, M_1, \dots, M_N)$ is large enough. This completes the proof of Theorem 1.1.

Proof of Lemma 3.3. It suffices to show that for $(j, \ell) \neq (k, \ell')$,

$$(3.6) \quad \limsup_{n \rightarrow \infty} \|U(t)\Phi_n^{\ell,j} U(t)\Phi_n^{\ell',k}\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} = 0.$$

When $(j, \ell) \neq (k, \ell')$, there are two possibilities:

- (1) $\limsup_{n \rightarrow \infty} \left(\frac{h_n^k}{h_n^j} + \frac{h_n^j}{h_n^k} \right) = \infty$,
- (2) $(h_n^j) = (h_n^k)$ and $\limsup_{n \rightarrow \infty} \frac{|t_n^{\ell,j} - t_n^{\ell',k}|}{(h_n^j)^\alpha} = \infty$.

More generally we will prove that if $\Psi_1, \Psi_2 \in L_t^q L_x^r$, then

$$\limsup_{n \rightarrow \infty} \left\| \frac{1}{(h_n^j)^{\frac{d}{2}}} \Psi_1\left(\frac{t - t_n^{\ell,j}}{(h_n^j)^\alpha}, \frac{x}{h_n^j}\right) \frac{1}{(h_n^k)^{\frac{d}{2}}} \Psi_2\left(\frac{t - t_n^{\ell',k}}{(h_n^k)^\alpha}, \frac{x}{h_n^k}\right) \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} = 0.$$

By density argument, it suffices to show this for $\Psi_1, \Psi_2 \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d)$. Using Hölder's inequality and scaling on space, we have

$$\begin{aligned} A_n &:= \left\| \frac{1}{(h_n^j)^{\frac{d}{2}}} \Psi_1\left(\frac{t - t_n^{\ell,j}}{(h_n^j)^\alpha}, \frac{x}{h_n^j}\right) \frac{1}{(h_n^k)^{\frac{d}{2}}} \Psi_2\left(\frac{t - t_n^{\ell',k}}{(h_n^k)^\alpha}, \frac{x}{h_n^k}\right) \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \\ &\leq \left\| \frac{1}{(h_n^j)^{\frac{d}{2} - \frac{d}{r}}} \Psi_1\left(\frac{t - t_n^{\ell,j}}{(h_n^j)^\alpha}, x\right) \right\|_{L_x^r} \left\| \frac{1}{(h_n^k)^{\frac{d}{2} - \frac{d}{r}}} \Psi_2\left(\frac{t - t_n^{\ell',k}}{(h_n^k)^\alpha}, x\right) \right\|_{L_x^r} \left\| \right\|_{L_t^{\frac{q}{2}}}. \end{aligned}$$

Then by time translation and scaling on time, we have

$$A_n \leq \left\| \left(\frac{h_n^j}{h_n^k} \right)^{\frac{\alpha}{q}} \Psi_1(t, x) \right\|_{L_x^r} \left\| \Psi_2\left(\left(\frac{h_n^j}{h_n^k} \right)^\alpha t - \frac{t_n^{\ell',k} - t_n^{\ell,j}}{(h_n^k)^\alpha}, x \right) \right\|_{L_x^r} \left\| \right\|_{L_t^{\frac{q}{2}}}.$$

Since the support in time of $\|\Psi_1(t, \cdot)\|_{L_x^r}$ is compact, from the above conditions (1) and (2) it readily follows that $\limsup_{n \rightarrow \infty} A_n = 0$. This completes the proof of Lemma 3.3. \square

4. NONLINEAR PROFILE DECOMPOSITION

In this section we prove Proposition 1.5 by making use of Theorem 1.1.

For simplicity of notations, we denote $\Gamma_n^j \phi^j$ by ϕ_n^j and $\Gamma_n^j \psi^j$ by ψ_n^j . First, we will show the forward implication. Fix $I = [a, b] \subset I_n$ for all n . We set $e_n^l = u_n - \sum_{j=1}^l \Gamma_n^j \psi^j - U(\cdot) \omega_n^l$, and

$$\|e_n^l\|_{[I]} := \|e_n^l\|_{C_t L_x^2(I \times \mathbb{R}^d)} + \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)}.$$

Since $\lim_l \limsup_n \|U(\cdot) \omega_n^l\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)} = 0$ and

$$\|u_n\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} \leq \sum_{j=1}^l \|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} + 1$$

for a large l , it suffices to show

$$(4.1) \quad \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e_n^l\|_{[I_n]} \rightarrow 0.$$

We write the equation for e_n^l in the following:

$$\begin{cases} i(e_n^l)_t + (-\Delta)^{\frac{\alpha}{2}} e_n^l = F(\sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l + e_n^l) - \sum_{j=1}^l F(\psi_n^j), \\ e_n^l(0, x) = \sum_{j=1}^l \phi_n^j(x) - \psi_n^j(0, x), \end{cases}$$

where $F(v) = (|x|^{-\alpha} * |v|^2)v$. Then Strichartz estimates give

$$(4.2) \quad \begin{aligned} \|e_n^l\|_{[I]} &\lesssim \|e_n^l(a, \cdot)\|_{L_x^2} \\ &+ \|F(\sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l + e_n^l) - F(\sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l)\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} \\ &+ \|F(\sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l) - \sum_{j=1}^l F(\psi_n^j)\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)}. \end{aligned}$$

To estimate each term on the right hand side, we use the orthogonality of nonlinear profile, in addition to the Hardy-Littlewood-Sobolev inequality. Denote the third term in (4.2) by

$$\beta_n^l := \|F(\sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l) - \sum_{j=1}^l F(\psi_n^j)\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)}.$$

Lemma 4.1. *There exist n_0, l_0 such that for all $n \geq n_0, l \geq l_0$,*

$$(4.3) \quad \sup_{l, n} \left\| \sum_{j=1}^l \psi_n^j + U(\cdot) \omega_n^l \right\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)} < \infty,$$

and

$$(4.4) \quad \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \beta_n^l \rightarrow 0.$$

Proof. First, we show (4.3). Since $\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|U(\cdot)\omega_n^l\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)} = 0$, it suffices to show

$$\left\| \sum_{j=1}^l \psi_n^j \right\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)} < \infty.$$

It follows from the small data global well-posedness that $\sum_{j=l_0}^\infty \|\psi_n^j\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^2 \leq \sum_{j=l_0}^\infty \|\phi_n^j\|_{L_x^2}^2 < 1$, for some large l_0 . Due to the orthogonality (3.5) and (3.6), for any l , we have

$$\begin{aligned} \left\| \sum_{j=1}^l \psi_n^j \right\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^2 &\leq \sum_{j=1}^l \|\psi_n^j\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^2 + o_n(1) \\ &\leq \sum_{j=1}^{l_0} \|\psi_n^j\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^2 + 2 \lesssim M \cdot l_0 + 2, \end{aligned}$$

where M is a uniform bound of $\{\|u_n^0\|_{L^2}^2\}$. For (4.4), we expand the cubical expressions $(F(\sum_l \cdot))$ and estimate

$$\begin{aligned} \beta_n^l &\leq \sum_{\{j_1=j_2=j_3\}^c} \|(|x|^{-\alpha} * (\psi_n^{j_1} \psi_n^{j_2})) \psi_n^{j_3}\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} \\ &\quad + \sum_{j_1, j_2} \|(|x|^{-\alpha} * (\psi_n^{j_1} U(\cdot) \omega_n^l) \psi_n^{j_2})\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} \\ &\quad + \sum_{j_1, j_2} \|(|x|^{-\alpha} * (\psi_n^{j_1} U(\cdot) \omega_n^l) \psi_n^{j_2})\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} + \|F(U(\cdot) \omega_n^l)\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)}. \end{aligned}$$

Since (q_\circ, r_\circ) is α -admissible, we can use the estimate

$$\|(|x|^{-\alpha} * (v_1 v_2)) v_3\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} \lesssim \prod_{1 \leq i \leq 3} \|v_i\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)},$$

to get

$$\begin{aligned} \beta_n^l &\lesssim \sum_{\{j_1=j_2=j_3\}^c} \prod_{1 \leq i \leq 3} \|\psi_n^{j_i}\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)} \\ &\quad + \|U(\cdot) \omega_n^l\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)} \left(\sum_{j_1, j_2} \prod_{i=1,2} \|\psi_n^{j_i}\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)} + \|U(\cdot) \omega_n^l\|_{L_t^{q_\circ} L_x^{r_\circ}(I \times \mathbb{R}^d)}^2 \right). \end{aligned}$$

Then from the orthogonality of nonlinear profiles (as like the proof of Lemma 3.3), (4.3) and

$$\limsup_{n \rightarrow \infty} \|U(t) \omega_n^l\|_{L_t^{q_\circ} L_x^{r_\circ}} = 0, \quad \text{for each } l$$

we conclude (4.4). \square

In order to handle the second term of (4.2), we first use Hölder's and the Hardy-Littlewood-Sobolev inequality to estimate

$$\begin{aligned} & \|F(\sum_{j=1}^l \psi_n^j + U(\cdot)\omega_n^l + e_n^l) - F(\sum_{j=1}^l \psi_n^j + U(\cdot)\omega_n^l)\|_{L_t^1 L_x^2(I \times \mathbb{R}^d)} \\ & \lesssim \sum_{k=1}^2 \left\| \sum_{j=1}^l \psi_n^j + U(\cdot)\omega_n^l \right\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)}^{3-k} \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)}^k. \end{aligned}$$

Substituting this into (4.2) and taking limsup, by Lemma 4.3 and (4.3) we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|e_n^l\|_{[I]} & \lesssim \limsup_{n \rightarrow \infty} \|e_n^l(a, \cdot)\|_{L^2} + \limsup_{n \rightarrow \infty} \sum_{j=1}^l \|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)}^2 \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)} \\ & \quad + \limsup_{n \rightarrow \infty} \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I \times \mathbb{R}^d)}^2. \end{aligned}$$

To handle remaining terms in the right hand side, we will divide interval I_n as in following lemma.

Lemma 4.2. *For given $\epsilon > 0$, there exist intervals I_n^1, \dots, I_n^ℓ such that $I_n = \cup_{i=1}^\ell I_n^i$ and*

$$\limsup_{n \rightarrow \infty} \sum_{j=1}^l \|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(I_n^i \times \mathbb{R}^d)} \leq \epsilon, \quad 1 \leq i \leq \ell.$$

Proof. The global well-posedness for small data and orthogonality give

$$\limsup_{n \rightarrow \infty} \left\| \sum_{j \geq \tilde{l}} \psi_n^j \right\|_{L_t^{q_0} L_x^{r_0}(I_n^i \times \mathbb{R}^d)}^2 \leq \limsup_{n \rightarrow \infty} \sum_{j \geq \tilde{l}} \|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(I_n^i \times \mathbb{R}^d)}^2 \leq \frac{\epsilon}{2}$$

for sufficiently large \tilde{l} . Let I^1 be maximal existence interval of ψ^1 . Since

$$\|\psi_n^1\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} = \|\psi^1\|_{L_t^{q_0} L_x^{r_0}((I_n + t_n^1) * (h_n^1)^\alpha \times \mathbb{R}^d)},$$

there exists $\tilde{I}^1 \subset I^1$ such that $\|\psi^1\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}^1 \times \mathbb{R}^d)} < \infty$ and $(I_n + t_n^1) * (h_n^1)^\alpha \subset \tilde{I}^1$. Hence we can find ℓ_1 and \tilde{I}_1^1 such that $\tilde{I}_1^1 = \cup_{i=1}^{\ell_1} \tilde{I}_i^1$ and $\|\psi^1\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_1^1 \times \mathbb{R}^d)} \leq \epsilon/2\tilde{l}$. Thus

$$\|\psi_n^1\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_{n,i}^1 \times \mathbb{R}^d)} < \epsilon/2\tilde{l},$$

where $\tilde{I}_{n,i}^1 = I_i^1/(h_n^1)^\alpha - t_n^1$. By repeating this argument we get ℓ_j and $\tilde{I}_{n,i}^j$, for $1 \leq j \leq \tilde{l}$, satisfying

$$\|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_{n,i}^j \times \mathbb{R}^d)} < \epsilon/2\tilde{l}.$$

Then by taking intersection of $\tilde{I}_{n,i}^j$ and I_n , we have $\{I_n^i\}_{i=1}^\ell$ with $\ell = \sum_{i=1}^{\tilde{l}} \ell_i$. \square

For $I = I_n^1$ we thus have up to a subsequence

$$\|e_n^l\|_{[I_n^1]} \lesssim \|e_n^l(0, \cdot)\|_{L^2} + \beta_n^l + \epsilon^2 \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(I_n^1 \times \mathbb{R}^d)} + \|e_n^l\|_{[I_n^1]}^2,$$

provided n is sufficiently large. By taking small $\epsilon > 0$ we get

$$\|e_n^l\|_{[I_n^1]} \lesssim \|e_n^l(0, \cdot)\|_{L^2} + \beta_n^l + \|e_n^l\|_{[I_n^1]}^2.$$

Since $\lim_l \limsup_n \|e_n^l(0, \cdot)\|_{L^2} = 0$, by continuity argument $\limsup_{l,n \rightarrow \infty} \|e_n^l\|_{[I_n^1]} = 0$. Particularly, this implies that $\limsup_{l,n \rightarrow \infty} \|e_n^l(b_n^1, \cdot)\|_{L_x^2} = 0$, where $I_n^1 = [a_n^1, b_n^1]$ and $a_n^1 = 0$. Then repeated arguments give $\lim_{n \rightarrow \infty} \|e_n^l\|_{[I_n^j]} \rightarrow 0$ as $l \rightarrow \infty$ for $1 \leq j \leq \ell$.

Now we show the implication (2) \rightarrow (1). Suppose that the statement is wrong. Then $\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} < \infty$ and there exists j_0 such that

$$\limsup_{n \rightarrow \infty} \|\psi_n^{j_0}\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)} = \infty.$$

By continuity, for given M , we have $\tilde{I}_n \subset I_n$ satisfying

$$\begin{aligned} M &< \limsup_{n \rightarrow \infty} \|\psi_n^{j_0}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}, \\ \limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{j=1}^l \|\psi_n^j\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} &< \infty. \end{aligned}$$

Then the implication (1) \rightarrow (2) gives $u_n = \sum_{j=1}^l \psi_n^j + U(\cdot)\omega_n^l + e_n^l$. Squaring this, we get

$$|u_n - U(\cdot)\omega_n^l - e_n^l|^2 - \operatorname{Re} \sum_{j_1 > j_2}^l \psi_n^{j_1} \overline{\psi_n^{j_2}} = \sum_{j=1}^l |\psi_n^j|^2.$$

Then Minkowski's inequality with $q, r \geq 2$ gives

$$\begin{aligned} &\left\| \left(\sum_{j=1}^l |\psi_n^j|^2 \right)^{\frac{1}{2}} \right\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 = \left\| \sum_{j=1}^l |\psi_n^j|^2 \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} \\ &= \left\| |u_n - U(\cdot)\omega_n^l - e_n^l|^2 - \operatorname{Re} \sum_{j_1 > j_2}^l \psi_n^{j_1} \overline{\psi_n^{j_2}} \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} \\ &\lesssim \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 + \|U(\cdot)\omega_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 + \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 \\ &\quad + \sum_{j_1 > j_2}^l \|\psi_n^{j_1} \overline{\psi_n^{j_2}}\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)}. \end{aligned}$$

Due to orthogonality, we obtain

$$\limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| \left(\sum_{j=1}^l |\psi_n^j|^2 \right)^{\frac{1}{2}} \right\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \lesssim \limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}.$$

On the other hand, we have

$$\begin{aligned} &\left\| |u_n - U(\cdot)\omega_n^l - e_n^l|^2 \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} \\ &\leq \left\| \sum_{j=1}^l |\psi_n^j|^2 \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} + \left\| \operatorname{Re} \sum_{j_1 > j_2}^l \psi_n^{j_1} \overline{\psi_n^{j_2}} \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)}. \end{aligned}$$

And we also have

$$\begin{aligned} & \left\| u_n - U(\cdot)\omega_n^l - e_n^l \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)}^{\frac{1}{2}} = \|u_n - U(\cdot)\omega_n^l - e_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \\ & \geq \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} - \|U(\cdot)\omega_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} - \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \\ & \leq \left(\left\| \sum_{j=1}^l |\psi_n^j|^2 \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} + \left\| \operatorname{Re} \sum_{j_1 > j_2}^l \psi_n^{j_1} \overline{\psi_n^{j_2}} \right\|_{L_t^{q_0/2} L_x^{r_0/2}(\tilde{I}_n \times \mathbb{R}^d)} \right)^{\frac{1}{2}} \\ & \quad + \|U(\cdot)\omega_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} + \|e_n^l\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \end{aligned}$$

and $\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \leq \limsup_{l \rightarrow \infty, n \rightarrow \infty} \|(\sum_{j=1}^l |\psi_n^j|^2)^{\frac{1}{2}}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}$ by orthogonality. So it follows that

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)} \approx \limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(\sum_{j=1}^l |\psi_n^j|^2)^{\frac{1}{2}}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}.$$

Therefore, we get

$$\begin{aligned} M^2 & < \limsup_{n \rightarrow \infty} \|\psi_n^{j_0}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 \leq \limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|(\sum_{j=1}^l |\psi_n^j|^2)^{\frac{1}{2}}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 \\ & \lesssim \limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n \times \mathbb{R}^d)}^2 \leq \limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}(I_n \times \mathbb{R}^d)}^2, \end{aligned}$$

which gives a contradiction by letting $M \rightarrow \infty$. This completes the proof.

5. BLOWUP PHENOMENA

In this section we provide the proofs of Theorem 1.6, 1.7, and Corollary 1.8

Proof of Theorem 1.6. By definition of δ_0 , there exist blowup solutions $\{u_n\}_{n=1}^\infty$ with initial data $\{u_{0,n}\}_{n=1}^\infty \subset L_x^2$ such that $\|u_{0,n}\| \searrow \delta_0$ as $n \rightarrow \infty$. By using time translation and scaling symmetry, we may assume that

$$\|u_n\|_{L_t^{q_0} L_x^{r_0}([0,1] \times \mathbb{R}^d)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Then we apply Theorem 1.1 to $\{u_{0,n}\}$ to get linear profiles $\{\phi^j, h_n^j, s_n^j\}$. From Proposition 1.5, we obtain nonlinear profiles $\{\psi^j\}$ associated with $\{\phi^j, h_n^j, s_n^j\}$.

Since $\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}([0,1] \times \mathbb{R}^d)} = \infty$, Proposition 1.5 says that there exists j_0 such that ψ^{j_0} blows up and so we have $\|\phi^{j_0}\|_{L^2} \geq \delta_0$. And by Theorem 1.1, we have

$$\|\phi^{j_0}\|_{L^2}^2 \leq \sum_{j \geq 1} \|\phi^j\|_{L^2}^2 \leq \limsup_{n \rightarrow \infty} \|u_{0,n}\|_{L^2}^2 = \delta_0^2$$

which implies

$$\|\psi^{j_0}(0, \cdot)\|_{L^2} = \|\phi^{j_0}\|_{L^2} \leq \delta_0.$$

Hence, $\|\phi^{j_0}\|_{L^2}$ should be δ_0 . For the proof of the second conclusion, we apply the above argument to the sequence $\{u(t_n)\}$.

Proof of Theorem 1.7. Let $u_n(t, x) = u(t + t_n, x)$. Then we have

$$\int_{\mathbb{R}^d} |u_n|^2 dx = \int_{\mathbb{R}^d} |u|^2 dx,$$

and

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}([0, T^* - t_n] \times \mathbb{R}^d)} = \limsup_{n \rightarrow \infty} \|u_n\|_{L_t^{q_0} L_x^{r_0}([-t_n, 0] \times \mathbb{R}^d)} = \infty.$$

Let $\{\phi^j, \psi^j, h_n^j, s_n^j\}$ be family of linear and nonlinear profiles associated with $\{u_n(0, \cdot)\}$ which are obtained in Theorem 1.1 and Proposition 1.5. We take the inverse of symmetry group (or we redefine $s_n^j := -\frac{s_n^j}{(h_n^j)^\alpha}$ and $h_n^j := \frac{1}{h_n^j}$). Then by Proposition 1.5 for $I_n = [0, T^* - t_n]$, there exists j_0 such that

$$\limsup_{n \rightarrow \infty} \|\psi^{j_0}\|_{L_t^{q_0} L_x^{r_0}(I_n^{j_0} \times \mathbb{R}^d)} = \infty,$$

where $I_n^{j_0} := [s_n^{j_0}, (T^* - t_n)/(h_n^{j_0})^\alpha + s_n^{j_0}]$.

Let $s^{j_0} := \limsup_{n \rightarrow \infty} s_n^{j_0}$. From Lemma A.1, we obtain $s^{j_0} \neq \infty$. Hence $s^{j_0} = -\infty$, or $s^{j_0} = 0$. If $s^{j_0} = -\infty$, then ψ^{j_0} blows up at T^{**} and $\limsup_{n \rightarrow \infty} (T^* - t_n)/(h_n^{j_0})^\alpha \geq T^{**}$. Applying the same argument to $\tilde{I}_n = [-t_n, 0]$, we get \tilde{j}_0 which satisfies

$$\limsup_{n \rightarrow \infty} \|\psi^{\tilde{j}_0}\|_{L_t^{q_0} L_x^{r_0}(\tilde{I}_n^{\tilde{j}_0} \times \mathbb{R}^d)} = \infty,$$

where $\tilde{I}_n^{\tilde{j}_0} := [(-t_n)/(h_n^{\tilde{j}_0})^\alpha + s_n^{\tilde{j}_0}, s_n^{\tilde{j}_0}]$. Since $\|u(0, x)\|_{L^2} < \sqrt{2}\delta_0$, there cannot be two blowup profiles. Hence \tilde{j}_0 should be j_0 . Therefore, from Lemma A.1, we get $s^{j_0} \neq -\infty$.

Now we have $s^{j_0} = 0$. Then Theorem 1.1 gives

$$(\Gamma_n^{j_0})^{-1} u_n(0, \cdot) = \phi^{j_0} + \sum_{j \neq j_0}^l (\Gamma_n^{j_0})^{-1} \Gamma_n^j \phi^j + (\Gamma_n^{j_0})^{-1} \omega_n^l.$$

Due to the orthogonality, $(\Gamma_n^{j_0})^{-1} \Gamma_n^j \phi^j \rightharpoonup 0$ weakly in L^2 as $n \rightarrow \infty$. And since $\limsup_{n \rightarrow \infty} \|\omega_n^l\|_{L_t^{q_0} L_x^{r_0}} \rightarrow 0$ as $l \rightarrow \infty$, the uniqueness of weak limit gives $(\Gamma_n^{j_0})^{-1} \omega_n^l \rightharpoonup 0$ weakly in L^2 . Hence we have

$$(\Gamma_n^{j_0})^{-1} u(t_n, \cdot) \rightharpoonup \phi^{j_0} \text{ weakly in } L^2.$$

Therefore, by taking $h_n = h_n^{j_0}$ and $\phi = \phi^{j_0}$, we see (1.3) and (1.4). This completes the proof of Theorem 1.7.

Proof of Corollary 1.8. By Theorem 1.7, there exists $\phi \in L_x^2$ such that $\|\phi\|_{L^2} \geq \delta_0$, and (1.3) and (1.4) hold. Hence we have for $R > 0$,

$$\limsup_{n \rightarrow \infty} (h_n)^d \int_{|x| \leq R} |u(t_n, h_n x)|^2 dx \geq \int_{|x| \leq R} |\phi|^2 dx.$$

After dilation, we get

$$\limsup_{n \rightarrow \infty} \int_{|x| \leq R h_n} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |\phi|^2 dx.$$

Since $\frac{(T^* - t_n)^{1/\alpha}}{\lambda(t_n)} \rightarrow 0$ as $t_n \rightarrow T^*$, we get $\frac{h_n}{\lambda(t_n)} \rightarrow 0$ and

$$\limsup_{n \rightarrow \infty} \int_{|x| \leq \lambda(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq R} |\phi|^2 dx.$$

Since $\int |\phi|^2 dx \geq \delta_0^2$, letting $R \rightarrow \infty$, we get (1.5).

APPENDIX A.

The local well-posedness of (1.1) is obtained in [7]. The well-posedness for a given asymptotic state is similar and fairly standard. We provide its proof for completeness.

Lemma A.1. *Given $g \in L^2(\mathbb{R}^d)$, there exists a positive T and a unique solution u to (1.1) such that $u \in C_t L_x^2([T, \infty) \times \mathbb{R}^d) \cap L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)$ and*

$$\|u(t) - U(t)g\|_{L_x^2} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Proof. We sketch the proof as the argument is rather standard. We define nonlinear mapping \mathcal{N} by

$$\mathcal{N}(v)(t) := i\lambda \int_t^\infty U(t-s)(|x|^{-\alpha} * |U(s)g + v(s)|^2)(U(s)g + v(s))ds$$

for v in Banach space $X = X_{T,\varepsilon}$ given by

$$X := \{v \in C_t L_x^2([T, \infty) \times \mathbb{R}^d) \cap L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d) :$$

$$\|v\|_{C_t L_x^2([T, \infty) \times \mathbb{R}^d)} + \|v\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)} \leq \varepsilon\}.$$

Using the Strichartz estimate (Lemma 2.1) and Christ-Kiselev lemma, one can get

$$\begin{aligned} \|\mathcal{N}(v)\|_{C_t L_x^2([T, \infty) \times \mathbb{R}^d)} + \|\mathcal{N}(v)\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)} &\lesssim \|U(s)g + v(s)\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)}^3 \\ &\lesssim (\|U(s)g\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)}^3 + \|v(s)\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)}^3). \end{aligned}$$

Since $\|U(s)g\|_{L_t^{q_0} L_x^{r_0}([T, \infty) \times \mathbb{R}^d)} \lesssim \|g\|_{L_x^2}$ by Lemma 2.1, \mathcal{N} becomes a self-mapping on X for sufficiently large T . Similarly one can easily prove that \mathcal{N} is a contraction mapping on X . Lastly the absolute continuity gives $\|v(t)\|_{L_x^2} \rightarrow 0$ as $t \rightarrow \infty$.

Now we write $u(t)$ as

$$u(t) = U(t)g + v(t).$$

Then $\|u(t) - U(t)g\|_{L_x^2} \rightarrow 0$ as $t \rightarrow \infty$. It remains to show that

$$(A.1) \quad u(\tau) = U(\tau - t)u(t) - i\lambda \int_t^\tau U(\tau - s)(|x|^{-\alpha} * |u|^2)u(s)ds.$$

In fact, since $v(\tau) = \mathcal{N}(v)(\tau)$, one can show that

$$v(\tau) = U(\tau - t)v(t) - i\lambda \int_t^\tau U(\tau - s)(|x|^{-\alpha} * |u|^2)u(s)ds.$$

Thus

$$u(\tau) = U(\tau)g + v(\tau) = U(\tau - t)(U(t)g + v(t)) - i\lambda \int_t^\tau U(\tau - s)(|x|^{-\alpha} * |u|^2)u(s)ds,$$

which yields (A.1). \square

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